Abstract

Axiom of Monotonicity is used along with Zermelo-Fraenkel set theory to derive Generalized Continuum Hypothesis. Axiom of Fusion is used to investigate the cardinality of the set of points in a unit interval.

Two Axioms to Extend ZF Theory

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1 INTRODUCTION

The purpose of this electronic announcement is to summarize some of the significant results that are to appear in two forthcoming paper publications on the foundations of set theory and give some extensions of those results [1, 2]. These papers define an axiomatic theory called *Intuitive Set Theory* (IST), in which Generalized Continuum Hypothesis (GCH) and Axiom of Choice (AC) are theorems. A crucial concept in IST is that of a *bonded set* with illusive elements in it, a notion somewhat like the quarks in particle physics. The introduction of bonded sets also makes it impossible to produce sets which are not Lebesgue measurable.

Reasoning about reason is obviously unreasonable, yet that is what we are forced to do when we consider the foundations of mathematics [3]. Accepting this as unavoidable, we add two axioms to Zermelo-Fraenkel (ZF) set theory with the hope that we will not, thereby, introduce contradictions in it. Central to the derivation of GCH in IST is the Axiom of Monotonicity (AM), which once stated makes the deduction of GCH almost immediate. Another axiom in IST called the Axiom of Fusion (AF), converts all sets of cardinality greater than $\aleph_0$ into impregnable bonded sets leaving us with essentially $\aleph_0$ to deal with. These two axioms we want to state clearly so that we may examine them critically. What follows is self-contained and does not need any reference to [1, 2].

2 AXIOM OF MONOTONICITY

Here is how Halmos explains [4] the generation of ω_1, the ordinal corresponding to $\aleph_1$ from ω.

> ... In this way we get successively ω, $\omega 2$, $\omega 3$, $\omega 4$, $\cdots$. An application of the axiom of substitution yields something that follows them all in the same sense in which ω follows the natural numbers; that something is ω^2. After that the whole thing starts over again: ω^2+1, ω^2+2, $\cdots$, $\omega^2+\omega$, $\omega^2+\omega+1$, $\omega^2+\omega+2$, $\cdots$, $\omega^2+\omega 2$, $\omega^2+\omega 2+1$, $\cdots$, $\omega^2+\omega 3$, $\cdots$, $\omega^2+\omega 4$, $\cdots$, $\omega^2 2$, $\cdots$, $\omega^2 3$, $\cdots$, ω^3, $\cdots$, ω^4, $\cdots$, ω^ω, $\cdots$, $\omega^{(\omega^\omega)}$, $\cdots$, $\omega^{(\omega^{(\omega^\omega)})}$, $\cdots\cdots$. The next one after

all this is ϵ_0; then come ϵ_0+1, ϵ_0+2, $\cdots$, $\epsilon_0+\omega$, $\cdots$, $\epsilon_0+\omega 2$, $\cdots$, $\epsilon_0+\omega^2$, $\cdots$, $\epsilon_0+\omega^\omega$, $\cdots$, $\epsilon_0 2$, $\cdots$, $\epsilon_0\omega$, $\cdots$, $\epsilon_0\omega^\omega$, $\cdots$, ϵ_0^2, $\cdots\cdots\cdots$.

This explanation, perhaps one of the best available, is satisfactory if we are interested only in understanding what transfinite numbers are. But, if we want to go beyond and investigate the properties of these numbers, then we have to look for more terse notations. Here is a solution that looks promising.

For positive integers m and n, we define an infinite sequence of operators as follows.

$$\begin{aligned} m\otimes^0 n &= mn, \\ m\otimes^k 1 &= m, \\ m\otimes^k n &= m\otimes^h [m\otimes^h [\cdots [m\otimes^h m]]], \end{aligned}$$

where the number of m's in the product is n and $h=k-1$. It is easy to see that

$$\begin{aligned} m\otimes^1 n &= m^n, \\ m\otimes^2 n &= m^{m^{\cdot^{\cdot^{\cdot^m}}}}, \end{aligned}$$

where the number of m's tilting forward is n. We can continue to expand the operators in this fashion further, but we will not do so, since it does not serve any purpose here. We use these operators for symbolizing the transfinite cardinals of Cantor.

We remove the restriction on m and n to be positive integers and claim that these operators are meaningful even when m and n take transfinite cardinal values. We go even further and assert that

$$\aleph_{\alpha+1} = \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha$$

The reasonableness of this equation can be judged from the fact that the ordinal corresponding to $\aleph_1$, can be written in the form

$$\begin{aligned} \omega_1 &= \{0,1,2,\cdots\omega,\cdots\omega^2,\cdots\omega^\omega,\cdots{}^\omega\omega,\cdots,\cdots,\cdots\} \\ &= \{0,1,2,\cdots\omega,\cdots\omega\otimes^0\omega,\cdots\omega\otimes^1\omega,\cdots\omega\otimes^2\omega,\cdots\omega\otimes^3\omega,\cdots,\cdots,\cdots\}. \end{aligned}$$

This can be verified easily from the description of ω_1 given by Halmos earlier. One more equation we will assert is that

$$2^{\aleph_\alpha} = 2\otimes^1\aleph_\alpha.$$

With these notations we can state the axiom that we are interested in.

Axiom 1 (Axiom of Montonicity) *$\aleph_{\alpha+1} = \aleph_\alpha\otimes^{\aleph_0}\aleph_\alpha$, and $2^{\aleph_\alpha} = 2\otimes^1\aleph_\alpha$. Further, if $m_1\le m_2$, $k_1\le k_2$, and $n_1\le n_2$, then $m_1\otimes^{k_1} n_1 \le m_2\otimes^{k_2} n_2$.*

Theorem 2.1 (Continuum Theorem) *$\aleph_{\alpha+1} = m\otimes^k\aleph_\alpha$ for finite $m>1, k>0$.*

Proof. A direct consequence of the axiom of monotonicity is that, for finite $m > 1$ and $k > 0$,

$$2^{\aleph_\alpha} = 2 \ \otimes^1 \aleph_\alpha \leq m \otimes^k \aleph_\alpha \leq \aleph_\alpha \otimes^{\aleph_0} \aleph_\alpha = \aleph_{\alpha+1}.$$

When we combine this with Cantor's result

$$\aleph_{\alpha+1} \leq 2^{\aleph_\alpha},$$

the theorem follows.

Theorem 2.2 (Generalized Continuum Hypothesis) $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

Proof. If we put $m = 2$, $k = 1$ in the Continuum Theorem, we get

$$\aleph_{\alpha+1} = 2 \ \otimes^1 \aleph_\alpha = 2^{\aleph_\alpha},$$

making GCH a theorem.

Theorem 2.3 (Axiom of Choice) *Given any set S of mutually disjoint nonempty sets, there is a set containing a single member from each element of S.*

Proof. CGH implies AC, and we have already proved GCH.

3 AXIOM OF FUSION

Before we can state the axiom of fusion, it is necessary to give a few definitions. The most significant definition here is that of a bonded set, a set from which no constituent element can be isolated. Just as no amount of energy can isolate a quark, so it is with a bonded set, no amount of ingenuity can separate an element from it. In the following, *points* and *elements* have the same meaning.

Bonded Set A set is a bonded set, if axiom of choice can choose only the set itself as a whole and not its constituent elements.

Figment An element of a bonded set.

Real element An element which is not a figment.

Real set A set which has only real elements. Note that a bonded set can be a real element.

Class A set which has sets as its elements.

Bonded class A class which has only bonded sets as its elements.

$2^{\aleph_\alpha}$ Power set of $\aleph_\alpha$.

$\binom{\aleph_\alpha}{\aleph_\alpha}$ *Combinatorial set* of $\aleph_\alpha$, defined as the class of all the subsets of $\aleph_\alpha$ of cardinality $\aleph_\alpha$.

Real Cardinality Cardinality of a real set.

R The class of *infinite recursive* subsets of positive integers, a class of cardinality $\aleph_0$.

x An element of the class R, which defines an infinite binary sequence and hence equivalent to a real point in the unit interval $(0, 1]$.

$(x|$ The cartesian product $x \times 2^{\aleph_\alpha}$, which we will call the *infinitesimal* x.

Microcosm The cartesian product $R \times 2^{\aleph_\alpha}$, considered as an adequate representation of all the points of the unit interval $(0, 1]$.

N An element of R, which defines an infinite binary sequence written leftwards and hence called a *supernatural number*.

$|N)$ The cartesian product $2^{\aleph_\alpha} \times N$, and hence called a *cosmic stretch*.

Macrocosm The cartesian product $2^{\aleph_\alpha} \times R$, considered as an adequate representation of all *counting numbers*, even those above supernatural numbers.

Using these definitions, we can now state the axiom of fusion.

Axiom 2 (Axiom of Fusion) $(0, 1] = \binom{\aleph_\alpha}{\aleph_\alpha} = R \times 2^{\aleph_\alpha}$, *where* $x \times 2^{\aleph_\alpha}$ *is a bonded set.*

The axiom of fusion says that the significant *combinatorial* part of the power set of $\aleph_\alpha$ consists of $\aleph_0$ infinitesimal bonded sets, each of cardinality $2^{\aleph_\alpha}$. Thus the *real cardinality* of the class $\binom{\aleph_\alpha}{\aleph_\alpha}$ is $\aleph_0$.
We define Intuitive Set Theory as the theory we get when the axioms of monotonicity and fusion are added to ZF theory.

Theorem 3.1 (Combinatorial Theorem) $\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}$.

Proof. A direct consequence of the axiom of fusion is that

$$2^{\aleph_\alpha} \leq \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Since, $\binom{\aleph_\alpha}{\aleph_\alpha}$ is a subset of $2^{\aleph_\alpha}$,

$$\binom{\aleph_\alpha}{\aleph_\alpha} \leq 2^{\aleph_\alpha},$$

and the theorem follows.

Theorem 3.2 (Unification Theorem) *All the three sequences*
$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$
$\aleph_0, 2^{\aleph_0}, 2^{\aleph_1}, 2^{\aleph_2}, \ldots$
$\aleph_0, \binom{\aleph_0}{\aleph_0}, \binom{\aleph_1}{\aleph_1}, \binom{\aleph_2}{\aleph_2}, \ldots$
represent the same series of cardinals.

Proof. The axiom of monotonicity shows that the first two are the same, and the axiom of fusion shows that the last two are same.

Cantor's theorem asserts that every model of ZF theory has to have cardinality greater than $\aleph_0$. On the other hand, Löwenheim-Skolem theorem (LS) says that there is a model of ZF theory, whose cardinality is $\aleph_0$. These two contradictory statements together is called Skolem Paradox.

Intuitive set theory provides a reasonable way to resolve the Skolem Paradox. We merely take the LS theorem as stating that the *real* cardinality of a model of IST need not be greater than $\aleph_0$.

In ZF theory, it is known that there are sets which are not Lebesgue measurable, but it has not been possible to date to construct such a set, without invoking the axiom of choice. The usual method is to choose exactly one element from each of the set $x \times 2^{\aleph_\alpha}$ we defined earlier, and show that the set thus created is not Lebesgue measurable. This method is obviously not possible in IST, since $x \times 2^{\aleph_\alpha}$ is a bonded set, and therefore the axiom of choice cannot be used to produce a nonmeasurable subset of $[0, 1]$. Hence, it would not be unreasonable to assert that there are no sets in IST which are not Lebesgue measurable.

4 CONCLUSION

From the definitions given above, it is obvious that there is a one-to-one correspondence between the points in $(0, 1]$ and the counting numbers. Hence, our statements about microcosm are equally applicable to macrocosm also. Further, it should be clear that intuitive set theory will suffice for scientists to investigate the phenomenal world, and classical set theory will be needed only by mathematicians who want to probe the complexities of the noumenal universe.

The two axioms given here allow us to visualize the unit interval $(0, 1]$ in a simple way. We can consider $(0, 1]$ as a graph with $\aleph_0$ edges and $\aleph_0$ nodes, each edge representing an infinitesimal with $2^{\aleph_\alpha}$ figments in it. Each node represents a *Dedekind cut* in the interval.

If the axioms of monotonicity and fusion do not produce any contradictions in IST, we can divide the statements of IST into four mutually exclusive categories: F is a *theorem*, if a proof exists for F, but not for $\overline{F}$. F is a *falsehood*, if a proof exists for $\overline{F}$, but not for F. F is an *introversion*, if a proof exists for $\overline{F}$ when F is assumed, and a proof for F exists when $\overline{F}$ is assumed. F is a *profundity*, if a proof exists for neither F nor $\overline{F}$, and it is not an introversion.

It is easy to see that an introversion cannot be chosen as an axiom, since it will surely create a contradiction in the theory. Gödel has shown that a consistency statement in any theory is an introversion. The conclusion is that even though we might believe in the consistency of a theory, we can never choose it as an axiom. Note that according to our definitions, generalized continuum hypothesis and axiom of choice are profundities in ZF theory, whereas they are theorems in IST.

The main problem of mathematics is to classify the *entire set* of formulas of IST into the four categories above. A great achievement of the twentieth century is the recognition that this is never possible.

References

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[4] P. R. Halmos, *Naive Set Theory*, D. Van Nostrand Company, New York, NY, 1960.