Sentient Arithmetic and Gödel's Theorems

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Abstract—Sentient Arithmetic is defined as an extention of Elementary Arithmetic with three more derivation rules and the Incompleteness Theorems are derived within it without using any metalanguage. It is shown that Consistency cannot be chosen as an axiom.

Keywords—Sentient Arithmetic; Consistency; Gödel's Theorems.

1. INTRODUCTION

Since the discussion here is about some of the subtle aspects of mathematical logic [1-3], a brief but complete definition of *Sentient Arithmetic* (SA) is given first. We follow the conventional form of definition of an axiomatic theory, except that axioms are listed as part of the derivation rules. The purpose of this paper is to show that it is possible to prove the incompleteness theorems of Gödel entirely within SA, without using any metalanguage.

1. Symbol Schema

1. *Logical symbols*

$$\vee \quad \wedge \quad \sim \quad , \quad) \quad (\quad \forall \quad \exists$$

Here, $\vee$ stands for *Or*, $\wedge$ for *And*, $\sim$ for *Complementation*, $\forall$ for *For All*, and $\exists$ for *There Exists*.

2. *Variable symbols*

$$x, x_1, x_2, \cdots, y, y_1, y_2, \cdots, z, z_1, z_2, \cdots$$

These symbols are similar to the variables of mathematics.

3. *Label symbols*

$$a, a_1, a_2, \ldots, b, b_1, b_2, \ldots, c, c_1, c_2, \cdots$$

These symbols are similar to the arbitrary constants of mathematics. Eventhough they are not available in conventional logic, we have a good reason for introducing them. By defining them here, we have the flexibility to dispense with the quantifier symbols when convenient: $\forall x P(x)$ can be written as $P(x)$ and $\exists x P(x)$ as $P(a)$.

4. *Constant symbols*

$$0, 0', 0'', 0''', \cdots$$

usually written as

$$0, 1, 2, 3, \cdots$$

5. *Equality symbol*

$$=$$

defined by the derivation rules given later.

6. *Arithmetical symbols*

$$0 \quad + \quad \times \quad '$$

The symbol $'$ is read as successor.

7. *Defined symbols*

New symbols can be defined in terms of the above symbols as done later in the text. Such definitions can be useful for creating new logical symbols and also for abbreviating long formulas and large numbers.

2. Term Schema

1. $0, a, x, a_1, x_1, a_2, x_2, \cdots, b, y, b_1, y_1, b_2, y_2, \cdots, c, z, c_1, z_1, c_2, z_2, \cdots$ are terms.
2. $(u)+(v)$, $(u)\times(v)$, $(u)'$ are terms if u and v are terms.

Note that the symbols u and v are used only to describe the schema.

3. Formula Schema

1. $u = v$ is a formula, if u and v are terms.
2. If p and q are formulas, then $(p)\vee(q)$, $(p)\wedge(q)$, and $\sim(p)$ are formulas. Note that the symbols p and q are used only to describe the schema.
3. $\forall s\ p(s)$ and $\exists s\ p(s)$ are formulas, if p is a formula and s is a variable in $p(s)$. The logical symbols $\forall$ and $\exists$ (called quantifiers) never occur by themselves. They are always accompanied by a variable and appears as $\forall s$ or $\exists s$. Note that s is used only to describe the schema.

4. Definition Schema

1. $F(x)$: We assume that the formulas of SA can be enumerated. The function $F(x)$ gives the x^{th} formula in the list. In the formation of the formulas we assume that more than one complementation at a time is not allowed, since it does not serve any purpose and will merely complicate our discussion. We will refer to $F(x)$ as the formula stored at address x. In the list, address 0 is reserved for a special formula G given later.
2. $\bar{x}$: The address at which $\bar{F}(x)$ is stored we call $\bar{x}$. Thus $F(\bar{x}) = \bar{F}(x)$. It is easy to see that $\bar{x}$ is a primitive recursive function of x, and $\bar{\bar{x}} = x$. Roughly, a function is recursive if it can be programmed.
3. $P(x, y)$: The primitive recursive predicate (a very long formula) which says that the formula $F(y)$ is a proof of the formula $F(x)$.
4. $D(x)$: An abbreviation for $\exists y\ P(x, y)$ which says that $F(x)$ can be derived. It is not a recursive predicate.
5. $F(x) \Rightarrow F(y)$: The same as $\bar{D}(x) + F(y)$. When the context makes it clear, we will use $+$ instead of $\vee$, as is common. Similarly we omit $\wedge$ whenever the omission is obvious.
6. $F(0)$: The formula $\sim \exists y P(0, y)$ is stored at address 0. $F(0)$ says that $F(0)$ cannot be derived. We will use the symbol G for $\sim \exists y P(0, y)$ and for uniformity $F(g)$ for $F(0)$. Note that G can also be written as $\bar{D}(g)$ and $\bar{G}$ as $D(g)$. Observe that keeping the formula $\sim \exists y P(0, y)$ at address 0 in no way affects the recursive nature of $F(x)$.
7. $F(c)$: The formula $\sim \exists x\ D(x)D(\bar{x})$ has to appear some where in our list, we call that address, c. We will use the symbol C for $\sim \exists x\ D(x)D(\bar{x})$. C says that it is impossible to derive both $F(x)$ and $\bar{F}(x)$. C is read as *consistency* and, $\bar{C}$ as *contradiction.*

5. Derivation Schema

In the following a single line in the statement of the rule means that it can be freely introduced anywhere in a derivation, in short, they are axioms. The $\top$ used here is a rotated turnstile symbol with the meaning that the following line can be derived from what preceeds. The meaning of $\perp$ should be obvious. Note that these symbols are used only to describe the schema.

1. *Sentential rules*

Commutation rule

a) $p + q$ b) pq

$\top$

$q + p$ $\qquad$ $\top$

qp

Distribution rule

a) $p(q + r)$
$\top\bot$
$pq + pr$

b) $p + qr$
$\top\bot$
$(p + q)(p + r)$

Identity rule

$p + \bar{C}$
$\top\bot$
p

Complementation rule

a) p
$\top$
p

b) $p\bar{p}$
$\top\bot$
$\bar{C}$

These derivation rules have been obtained from the definition of boolean algebra. The omission of the axiom $p + \bar{p}$ is intentional here, since SA has no use for the law of the excluded middle.

Detachment rule

p
$p \Rightarrow q$
$\top$
q

2. *Predicate rule*

a) $\forall x\ p(x)$
$\top$
$\exists x\ p(x)$

b) $p(u)$
$\top$
$\exists x\ p(x)$

3. *Equality rule*

a) $u = u$

b) $u = v$
$\top$
$p(u, u) \Rightarrow p(u, v)$

4. *Peano rule*

a) $\sim \exists x\ (x' = 0)$

b) $u' = v'$
$\top$
$u = v$

c) $p(0)$
$\forall x\ [p(x) \Rightarrow p(x')]$
$\top$
$\forall x\ p(x)$

5. *Addition rule*

$u + 0 = u$ $\qquad$ $u + v' = (u + v)'$

6. *Multiplication rule*

$u \cdot 0 = 0$ $\qquad$ $u \cdot v' = u \cdot v + u$

7. *Sentient rules*

Validity rule: This rule essentially gives a syntactic definition of *truth*.

$D(u)$
$\top$
$F(u)$

Introspection rule: This rule says that if you have a legitimate derivation of $F(u)$ visibly in front of you, you can conclude that $D(u)$ is true.

$$\vdots$$
$$F(u)$$
$$\top$$
$$D(u)$$

Contradiction rule: This rule says that any formula that leads to a contradiction cannot be derived.

$F(u)$ ∘ assumption

$$\vdots$$
$$\bar{C}$$
$$\top$$
$$C \Rightarrow \bar{D}(u)$$

It is legitimate to use both the validity rule and the introspection rule under the assumption of the contradiction rule. This rule we may call *no-proof by contradiction.* The usual *proof by contradiction* is not allowed in our theory.

The schemas given above define Sentient Arithmetic. If we omit derivation rules 7 we get the Elementary Arithmetic of Gödel. There are no seperate axioms in SA, they are embedded in the derivation rules. A theorem in SA is the last line in a derivation. A derivation in which the last line is $\bar{C}$ is called a *virus*.

2. GÖDEL'S THEOREMS

Using the definitions given above, we can prove the incompleteness of SA entirely within SA.

FIRST INCOMPLETENESS THEOREM. $C \Rightarrow \bar{D}(g)\bar{D}(\bar{g})$

Proof of $C \Rightarrow \bar{D}(g)$

1. G ∘ assumption
2. $D(g)$ ∘ introspection rule on line 1
3. $\bar{D}(g)$ ∘ definition of G at line 1
4. $\bar{C}$ ∘ from lines 2 and 3

$\top$

$C \Rightarrow \bar{D}(g)$ ∘ contradiction rule

Proof of $C \Rightarrow \bar{D}(\bar{g})$

1. $\bar{G}$ ∘ assumption
2. $D(g)$ ∘ definition of $\bar{G}$ at line 1
3. G ∘ applying validity rule on line 2
4. $\bar{C}$ ∘ from lines 1 and 3

$\top$

$C \Rightarrow \bar{D}(\bar{g})$ ∘ contradiction rule

First Incompleteness Theorem immediately follows. □

SECOND INCOMPLETENESS THEOREM. $C \Rightarrow \bar{D}(c)\bar{D}(\bar{c})$

Proof of $C \Rightarrow \bar{D}(c)$

1. C ∘ assumption
2. $C \Rightarrow \bar{D}(g)$ ∘ first incompleteness theorem
3. $\bar{D}(g)$ ∘ detachment rule on lines 1 and 2
4. G ∘ $\bar{D}(g)$ at line 3 is the definition of G

5. $D(g)$ ○ applying introspection rule on lines 1 and 4
6. $\bar{C}$ ○ from lines 3 and 5

$\top$

$C \Rightarrow \bar{D}(c)$ ○ contradiction rule

Proof of $C \Rightarrow \bar{D}(\bar{c})$

1. $\bar{C}$ ○ assumption

$\top$

$C \Rightarrow \bar{D}(\bar{c})$ ○ contradiction rule

Second Incompleteness Theorem immediately follows. □

3. CONCLUSION

From our derivations it should be clear that SA abhors a contradiction. Then the question arises, why we should not introduce consistency itself as an axiom of SA. If we do that, unfortunately, viruses invade SA and it collapses, as shown by the argument below.

1. C ○ new axiom introduced
2. $D(c)$ ○ applying introspection rule on line 1
3. $C \Rightarrow \bar{D}(c)$ ○ second incompleteness theorem
4. $\bar{D}(c)$ ○ detachment rule on lines 1 and 3

$\top$

$\bar{C}$ ○ from lines 2 and 4

The conclusion is that C cannot be introduced as an axiom of SA. We may define $F(x)$ as a metastatement of SA, if both $F(x)$ and $\bar{F}(x)$ leads to a contradiction. Obviously, a metastatement cannot be chosen as an axiom of SA.

If what we have discussed here is sensible, we can divide the statements in SA into four classes. F is a *contradictory statement* if a derivation exists for both F and $\bar{F}$. F is an *arithmetical statement* if a derivation exists for either F or $\bar{F}$, but not both. F is a *mystic statement* if a derivation exists for neither F nor $\bar{F}$ and it is not a metastatement. As stated above, F is a *metastatement* if a derivation exists for $\bar{F}$ when F is assumed and a derivation for F exists when $\bar{F}$ is assumed.

To judge the quality of an axiomatic theory we make the following definitions. A theory is *sound* if there are no contradictory statements in it. A sound theory is *rational* if there are no mystic statements in it. We can now state the belief and hope that can be entertained about Sentient Arithmetic, in terms of these concepts.

Arithmetical Faith: *SA is sound*

Arithmetical Hope: *SA is rational*

Our discussion here clearly shows that it will be fatal for SA to convert the faith into an axiom and unrealistic to take the hope as a fact.

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