

Abstract

Ackermann functions are used recursively to define the transfinite cardinals of Cantor. Continuum Hypothesis and Axiom of Choice are derived from the definition. An axiom which splits the unit interval into infinitesimals is stated. Using illustrations, the tutorial visualizes the resulting set theory.

Keywords—Continuum Hypothesis, Axiom of Choice, Visualization.

Intuitive Set Theory: A Tutorial

K. K. Nambiar

Copyright © 2000

Last Revision Date: May 9, 2001

Table of Contents

- 1. INTRODUCTION**
- 2. AXIOM OF MONOTONICITY**
- 3. AXIOM OF FUSION**
- 4. UNIVERSAL NUMBER SYSTEM**
- 5. CONCLUSION**
Solutions to Exercises

1. INTRODUCTION

Some explanation is in order to justify the writing of yet another tutorial on set theory, which has been a subject of in-depth research papers for more than a century. One of the reasons for writing this expository article is my belief that I have defined an axiomatic theory which should be understandable to any well motivated novice, with no extensive background in any of the sophisticated areas of mathematics. Another reason is that the theory described here satisfies the Continuum Hypothesis, thereby, allowing us to have a simple visualization of the transfinite cardinals of Cantor. As explained in an earlier paper, *Intuitive Set Theory* (IST), the subject matter of this tutorial, is the theory we get when we add Axiom of Monotonicity and Axiom of Fusion to Zermelo-Fraenkel set theory [1]. The two axioms are explained later in this paper.

Some of the basic facts of set theory, we want to state first, but to begin with, let us recognize that for the set theorist, every natural number is a set, for example, the integer 5 is considered as the set

$$\{0, 1, 2, 3, 4\}.$$

Adopting this notation, the entire set of natural numbers can be written as

the sets

$$\begin{aligned}\{\} &= 0 \\ \{0\} &= 1 \\ \{0, 1\} &= 2 \\ \{0, 1, 2\} &= 3 \\ &\vdots\end{aligned}$$

The advantage with this method is that we get an elegant way of defining \aleph_0 , the first transfinite cardinal of Cantor, as

$$\{0, 1, 2, 3, \dots\},$$

and we will see later that we can define the higher cardinals also in this fashion.

An important characteristic of a set is its size or *cardinality*. Two sets are said to have the same cardinality, if a one-to-one correspondence can be set up between them. The set of all subsets of a set S is called the *powerset* of S . Cantor has shown (diagonal procedure) that the powerset of S will always have greater cardinality than the set S , even when S is an infinite

set. An important consequence of this is that we can without end construct bigger and bigger sets,

$$2^{\aleph_0}, 2^{2^{\aleph_0}}, 2^{2^{2^{\aleph_0}}}, \dots$$

and hence in set theory we cannot have a set which has the highest cardinality. A disappointing consequence is that we cannot have a universal collection as part of set theory and such a collection will always have to be outside the set theory. A significant set, as important as 2^{\aleph_α} is the set of all subsets of \aleph_α of cardinality \aleph_α . If we call this set $\binom{\aleph_\alpha}{\aleph_\alpha}$, it can be shown, **as done in the sequel**, that

$$\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}.$$

It is known in recursive function theory that the cardinality of infinite recursive subsets of natural numbers is \aleph_0 . We will have occasion to use this fact later.

2. AXIOM OF MONOTONICITY

Consider the following question: If somebody, we will call the person P , travels absolutely straight one meter per second indefinitely, where will P be eventually, and how long will it be, before P reaches there. The obvious unsatisfactory answer is that P will soon get out of sight and we will never know what happened to P after that. This answer is very disturbing to a mathematician and therefore he reformulates the question. In the modified form, P does not travel at uniform velocity, instead, travels the first meter in half a second and from then on, every meter takes only half the time of the previous meter. Now the question is, where will P be in exactly one second. Even though Cantor did not formulate the problem in this fashion, it is indirectly this question that Cantor faced, when he was developing his set theory [2]. Withstanding considerable antagonism from his peers, Cantor tells us that it would not be unreasonable to say that P has reached a distance \aleph_0 from the origin. To that, we can add with hindsight that the highest velocity reached by P is

$$2^{\{0,1,2,3,\dots\}} = 2^{\aleph_0}.$$

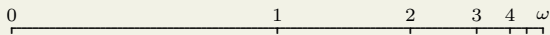
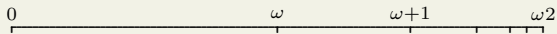
From the notation here, it should be clear that 2^{\aleph_0} can also be considered as the power set of \aleph_0 . Having whetted his appetite by the discovery of \aleph_0 and 2^{\aleph_0} , Cantor increased the velocity of P even more than exponential, and went on to discover

$$\aleph_1, 2^{\aleph_1}, \aleph_2, 2^{\aleph_2}, \dots$$

and finally asked the question: is $2^{\aleph_0} = \aleph_1$? By this time, the “heaven” reached by Cantor by his travel through the infinities was so enticing for the mathematicians, they raised the more general question: is $2^{\aleph_\alpha} = \aleph_{\alpha+1}$? This question is considered as one of the most important in mathematics today, IST answers the question in the affirmative.

P 's sojourn in Cantor's heaven is what set theory is all about. Unfortunately, we cannot talk about the distance or the speed with which P traveled in any natural language, and hence what we do here is to give the travelogue as simply as possible in terms of suggestive figures. These figures represent an infinite sequence of unit intervals.

To understand the sequence of figures from Fig. P_a to Fig. P_ω , we have to talk about *explosive operators*, which is essentially the same as the well-known **Ackermann functions**.

Fig. P_a . Set ω Fig. P_b . Set ω^2 Fig. P_0 . Set ω_0 Fig. P_1 . Set ω_1

⋮

Fig. P_ω . Set \aleph_1

Explosive Operators

For positive integers m and n , we define an infinite sequence of operators as follows.

$$m \otimes^0 n = mn,$$

$$m \otimes^k 1 = m,$$

$$m \otimes^k n = m \otimes^h [m \otimes^h [\dots [m \otimes^h m]]],$$

where the number of m 's in the product is n and $h = k - 1$. It is easy to see that

$$m \otimes^1 n = m^n,$$

$$m \otimes^2 n = m^{m^{\dots^m}},$$

where the number of m 's tilting forward is n . We can continue to expand the operators in this fashion further, straining our currently available notations, but we will not do so, since it does not serve any purpose here. We have called \otimes^k , explosive operators for a good reason. Some idea about the complexity of \otimes^k can be obtained, if one attempts to calculate the number

represented by $2 \otimes^3 4$. A detailed evaluation shows that

$$2 \otimes^3 4 = 2^{2^{\dots^2}},$$

the total number of 2's tilting forward being 65536. We use these operators for symbolizing the transfinite cardinals of Cantor [3].

We remove the restriction on m and n to be positive integers and claim that these operators are meaningful even when m and n take transfinite cardinal values. We go even further and assert that

$$\aleph_{\alpha+1} = \aleph_{\alpha} \otimes^{\aleph_0} \aleph_{\alpha}.$$

The reasonableness of this equation can be judged from the fact that \aleph_1 can be written in the form

$$\begin{aligned} \aleph_1 &= \{0, 1, 2, \dots, \omega, \dots, \omega^2, \dots, \omega^\omega, \dots, {}^\omega\omega, \dots, \dots, \dots\} \\ &= \{0, 1, 2, \dots, \omega, \dots, \omega \otimes^0 \omega, \dots, \omega \otimes^1 \omega, \dots, \omega \otimes^2 \omega, \dots, \dots, \dots\}. \end{aligned}$$

One more equation we will assert is that the powerset of \aleph_{α} ,

$$2^{\aleph_{\alpha}} = 2 \otimes^1 \aleph_{\alpha}.$$

It is easy to explain the sojourn of P in terms of the operators \otimes^k . In Fig. P_a are marked, the elements of $\omega = \aleph_0$, the natural numbers. The first half of Fig. P_b is obtained by contracting Fig. P_a to half its size, and the second half are marked the elements of $\omega 2$ above ω . The first half of Fig. P_0 is obtained by contracting Fig. P_b to half its size, and the second half are marked the elements of ω^2 above $\omega 2$. By repeating this process indefinitely, we get the infinite sequence of figures, finally ending up with Fig. P_ω . The heavy line of Fig. P_ω indicates that even the smallest of intervals in it anywhere, contains an infinite number of ordinals.

Travel from 0 to \aleph_0

Fig. P_a explains the travel of P from 0 to \aleph_0 . To prevent P from going out of sight with exponential speed, we contract the space exponentially every time P takes a step. The distance reached by P at every step is marked in the figure. The distance \aleph_0 appears at the end of the unit interval.

Travel from \aleph_0 to \aleph_1

To reach \aleph_1 , P travels the distance \aleph_0 in half-a-second and then steps through the significant ordinals $\omega, \omega 2, \omega^2, \omega^\omega, \dots$ increasing the speed expo-

sively at every step. To prevent P from going out of sight with explosive speed, we contract the space implosively every time P takes a step. At the end of exactly one second P reaches \aleph_1 , marked at the end of the unit interval in Fig. P_ω .

Travel from \aleph_0 to Ω

To reach Ω , the Absolute Infinity, P travels the distance \aleph_0 in half-a-second and then steps through the higher cardinals $\aleph_1, \aleph_2, \aleph_3, \dots$ increasing the speed explosively at every step. To prevent P from going out of sight with explosive speed, we contract the space implosively every time P takes a step. At the end of exactly one second P reaches Ω , which unfortunately is not a cardinal, as shown by Cantor. In a sense, P loses identity and existence, on arrival at Ω .

Cantor always wanted his Continuum Hypothesis, $2^{\aleph_0} = \aleph_1$, to be true in his set theory. We now **introduce an axiom** to accomplish this, and even more [4].

Axiom 1 (Axiom of Monotonicity)

$\aleph_{\alpha+1} = \aleph_{\alpha} \otimes^{\aleph_0} \aleph_{\alpha}$, and $2^{\aleph_{\alpha}} = 2 \otimes^1 \aleph_{\alpha}$. Further, if $m_1 \leq m_2$, $k_1 \leq k_2$, and $n_1 \leq n_2$, then $m_1 \otimes^{k_1} n_1 \leq m_2 \otimes^{k_2} n_2$.

From this axiom a significant theorem follows.

Theorem 2.1 (Continuum Theorem)

$\aleph_{\alpha+1} = m \otimes^k \aleph_{\alpha}$, for finite $m > 1, k > 0$.

Proof. A direct consequence of the axiom of monotonicity is that, for finite $m > 1$ and $k > 0$,

$$2^{\aleph_{\alpha}} = 2 \otimes^1 \aleph_{\alpha} \leq m \otimes^k \aleph_{\alpha} \leq \aleph_{\alpha} \otimes^{\aleph_0} \aleph_{\alpha} = \aleph_{\alpha+1}.$$

When we combine this with Cantor's result

$$\aleph_{\alpha+1} \leq 2^{\aleph_{\alpha}},$$

the theorem follows.

Theorem 2.2 (Generalized Continuum Hypothesis)

$\aleph_{\alpha+1} = 2^{\aleph_{\alpha}}$.

Proof. If we put $m = 2$, $k = 1$ in the Continuum Theorem, we get

$$\aleph_{\alpha+1} = 2 \otimes^1 \aleph_{\alpha} = 2^{\aleph_{\alpha}},$$

making GCH a theorem.

Theorem 2.3 (Axiom of Choice) *Given any set S of mutually disjoint nonempty sets, there is a set containing a single member from each element of S .*

Proof. GCH implies AC, and we have already proved GCH.

3. AXIOM OF FUSION

From the contracted pictures of P 's travels all the way to Ω in the mathematical universe, it is clear that the entire set of ordinals can be marked within a unit interval. From this it follows that the interval $[0, 1]$ can be used to represent all the ordinals of Cantor. Now we want to mark the ordinals in $[0, 1]$ rearranged independent of P 's travels. In the sequel, the terms *points* and *elements* are used interchangeably.

Note, as an example, that the infinite sequence $.010 * * * * \dots$ can be used to represent the interval $(.25, .375]$, if we accept certain assumptions about the representation:

The initial binary string, $.010 = .25$, represents the initial point of the interval.

The length of the binary string, 3 in our case, decides the length of the interval as $2^{-3} = .125$.

Every $*$ in the infinite $*$ -string can be substituted by a 0 or 1, to create 2^{\aleph_α} points in the interval.

We will accept the fact that a *nonterminating* binary sequence,

$$x = .bbbb\dots,$$

or equivalently, an infinite recursive subset of positive integers, can be used *uniquely* to represent a real number in the interval $(0, 1]$.

Consider as an example, the infinite binary sequence $.101010\dots ****\dots$. Using some freedom in terminology, we can say that this represents an *infinitesimal* of length \aleph_0^{-1} located at $\frac{2}{3}$ and the infinitesimal contains 2^{\aleph_α} points in it. The length of the infinitesimal is \aleph_0^{-1} because of the fact that the cardinality of R , the class of infinite recursive subsets of

positive integers is only \aleph_0 .

Since an infinite sequence is a precise form of specifying a number, which cannot be improved any further, we can claim that the infinitesimals are sets from which no element can be pried out. More precisely, we can say that, not even the axiom of choice can choose an element from an infinitesimal. We will call a set from which the **axiom of choice cannot choose**, a *bonded set*, and the elements in it *figments*. The cardinality we get when we ignore figments and take a bonded set as a single element, we will call *virtual cardinality*. The whole idea is made clear by the following axiom [1].

Axiom 2 (Axiom of Fusion) $(0, 1] = \binom{\aleph_\alpha}{\aleph_\alpha} = R \times 2^{\aleph_\alpha}$, where $x \times 2^{\aleph_\alpha}$ is a bonded class.

The axiom of fusion says that $(0, 1]$ is a class of bonded sets, called infinitesimals. Further, the significant *combinatorial* part of the power set of \aleph_α consists of \aleph_0 bonded sets, each of cardinality 2^{\aleph_α} . Thus the *virtual cardinality* of $\binom{\aleph_\alpha}{\aleph_\alpha}$ is \aleph_0 .

We define Intuitive Set Theory as the theory we get when the axioms of monotonicity and fusion are added to Zermelo-Fraenkel set theory (ZF).

Theorem 3.1 (Combinatorial Theorem) $\binom{\aleph_\alpha}{\aleph_\alpha} = 2^{\aleph_\alpha}$.

Proof. A direct consequence of the axiom of fusion is that

$$2^{\aleph_\alpha} \leq \binom{\aleph_\alpha}{\aleph_\alpha}.$$

Since, $\binom{\aleph_\alpha}{\aleph_\alpha}$ is a subset of 2^{\aleph_α} ,

$$\binom{\aleph_\alpha}{\aleph_\alpha} \leq 2^{\aleph_\alpha},$$

and the theorem follows.

Theorem 3.2 (Unification Theorem) *All the three sequences*

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \dots$$

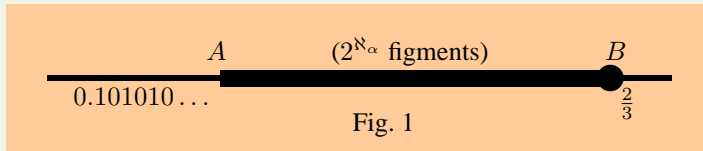
$$\aleph_0, 2^{\aleph_0}, 2^{\aleph_1}, 2^{\aleph_2}, \dots$$

$$\aleph_0, \binom{\aleph_0}{\aleph_0}, \binom{\aleph_1}{\aleph_1}, \binom{\aleph_2}{\aleph_2}, \dots$$

represent the same series of cardinals.

Proof. The axiom of monotonicity shows that the first two are the same, and the axiom of fusion shows that the last two are same.

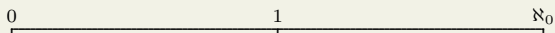
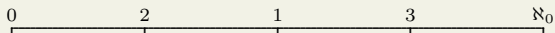
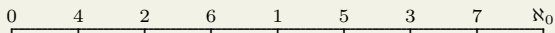
Fig. 1 should help to build up a mental picture for an infinitesimal. We want to imagine how an infinitesimal part of a unit interval looks like, when magnified \aleph_0 times. The heavy line in the figure is, perhaps, as good a representation as any for an infinitesimal, and we can imagine that \aleph_0 such infinitesimals constitute a unit interval. The age-old question about a number on the real line is, whether we should consider it as a tiny *iron filing* or as a *steel ball*. According to our view here, it is both. The line (A, B) in the figure is the filing and B is the ball, with the clear understanding that these are only figments of our imagination and can never be palpable.



The binary sequence $0.101010\dots$ shown in the figure indicates that the infinitesimal in our visualization corresponds to the number $\frac{2}{3}$ in the unit interval.

If we accept the visualization of an infinitesimal, we can define an *infinitesimal graph*, to represent the whole unit interval. In the Cantorian tradition, we define the infinitesimal graph G_{\aleph_0} as the infinite sequence of graphs shown as Figures $\{G_1, G_3, G_7, \dots\}$. Note that the graph G_k has k nodes between the nodes 0 and \aleph_0 , labeled 1 to k . We will take it as axiomatic that the graph G_{\aleph_0} shown in Fig. G_{\aleph_0} has \aleph_0 nodes and \aleph_0 edges, and also that each infinitesimal consists of an edge and a node. In the graphs, nodes are unconventionally drawn as vertical lines for clarity, and also in deference to Dedekind whose *cut* it represents.

Cantor's theorem asserts that every model of Zermelo-Fraenkel set the-

Fig. G_1 Fig. G_3 Fig. G_7 \vdots Fig. G_{\aleph_0}

ory has to have cardinality greater than \aleph_0 . On the other hand, Löwenheim-Skolem theorem (LS) says that there is a model of ZF theory, whose cardinality is \aleph_0 . These two statements together is called Skolem Paradox.

Intuitive set theory provides a reasonable way to resolve the Skolem Paradox. We merely take the LS theorem as stating that the *virtual* cardinality of a model of IST need not be greater than \aleph_0 . Clearly, the Upward Löwenheim-Skolem theorem also cannot raise any paradox in IST.

In measure theory, it is known that there are sets which are not Lebesgue measurable, but it has not been possible to date to construct such a set, without invoking the axiom of choice. The usual method is to choose exactly one element from each of the set $x \times 2^{\aleph_\alpha}$ we defined earlier, and show that the set thus created is not Lebesgue measurable. This method is obviously not possible in IST, since $x \times 2^{\aleph_\alpha}$ is a bonded set. Thus, it would not be unreasonable to assert that there are no sets in IST which are not Lebesgue measurable.

4. UNIVERSAL NUMBER SYSTEM

Without losing generality, we will restrict ourselves to binary number system. A number is normally defined as a string

$$\dots 000 \ast \ast \ast \dots \ast \ast \ast . \ast \ast \ast \dots$$

where the string on the left side of the binary point, has to be all zeroes eventually and the \ast s can take the values either 1 or 0. A slight extension of the 2s complement number system, extensively used in computer science, allows us to conclude that a string of the form

$$\dots 111 \ast \ast \ast \dots \ast \ast \ast . \ast \ast \ast \dots,$$

where the left-string ends up in 1s finally, can be used to represent negative numbers.

We define *universal number system* (UNS) as the system we get, when we allow infinite strings on both sides of the binary point, without any restriction. Given a universal number a , the number we get when we flip the two-way infinite string around the binary point, we will write as a^F . Given, a universal number a , the number we get by interchanging the 0s and 1s in it, we will write as $-a$. From our definition, it should be clear that the binary number system is a special case of the universal number system.

As mentioned earlier, every binary string of the form $a = \dots 000.***\dots$, represents a number in the interval $[0, 1]$. Some examples of a s and the corresponding a^F s are given below.

$$\begin{array}{ll} a_0 = \dots 000.000\dots = 0 & a_0^F = \dots 000.000\dots = 0 \\ a_1 = \dots 001.000\dots = 1 & a_1^F = \dots 000.100\dots = \frac{1}{2} \\ a_2 = \dots 010.000\dots = 2 & a_2^F = \dots 000.010\dots = \frac{1}{4} \\ a_3 = \dots 011.000\dots = 3 & a_3^F = \dots 000.110\dots = \frac{3}{4} \\ a_4 = \dots 100.000\dots = 4 & a_4^F = \dots 000.001\dots = \frac{1}{8} \\ a_5 = \dots 101.000\dots = 5 & a_5^F = \dots 000.101\dots = \frac{5}{8} \end{array}$$

The rationale behind the locations and names of the nodes in Fig. G_{\aleph_0} should be clear from the above values.

Consider the transcendental number $\frac{\pi}{4}$ in the interval $[0, 1]$.

$$\frac{\pi}{4} = \dots 000.11001000110\dots$$

and

$$\left(\frac{\pi}{4}\right)^F = \dots 01100010011.000\dots$$

which gives the appearance of a number above all natural numbers. For

this reason, we will call it a *supernatural number*, of course, it is no more supernatural than the transcendental number is transcendental. From this example, it should be clear that corresponding to every transcendental number in the interval $[0, 1]$, there is a supernatural number.

Our discussion shows that the supernatural numbers can be used to represent the entire set of transfinite ordinals. Just as an infinitesimal bonded set is attached with a number in the interval $(0, 1)$, an infinite *unreachable stretch* is attached with every supernatural number. Here, the word “stretch” is used as an aid to visualize an infinite set of points spread over an infinite line, but it is to be considered as an element, just as a bonded set is.

5. CONCLUSION

If the two axioms introduced here, do not produce any contradictions in ZF theory, IST produces a simple picture for visualizing all the ordinals of Cantor. The two axioms split the interval $[0, 1]$ into \aleph_0 bonded sets, each set containing \aleph_α figments. We consider the bonded sets as a single entity with figments in it, which even the axiom of choice cannot access.

If we accept the transfinite cardinals, certain other uncomfortable con-

clusions follow. Consider P shuttling back and forth within the unit interval in Fig. G_{\aleph_0} as follows: P travels from 0 to 1 in 2^{-1} second, turns around and goes from 1 to 2 in 2^{-2} second, turns around and goes from 2 to 3 in 2^{-3} second and so on. Where will P be in exactly one second? The inevitable conclusion is that P occupies the entire unit interval at the end of one second. Note that this omnipresence is in the strict sense and not in the quantum mechanical probabilistic sense. We can clearly see that the study of transfinite cardinals do lead us to philosophy, as Cantor had always claimed.

We will end with a cautionary note. There is ambiguity in the literature regarding the exponential notation 2^ω , since it has been used to represent both \aleph_0 and 2^{\aleph_0} . While this poses no problem in the hands of an expert, it would be preferable, if 2^ω is distinguished from 2^{\aleph_0} . A possible solution here is to use an old terminology and visualize ordinals as *potential infinities* and cardinals as *completed infinities*. In any case, in this paper the distinction must be kept clear, for example, ω^ω is not the same as $\aleph_0^{\aleph_0}$. Finally, note that this paper is only concerned with cardinals.

References

- [1] K. K. Nambiar, *Intuitive Set Theory*, Computers and Mathematics with Applications **39** (1999), no. 1-2, 183–185. [3](#), [16](#)
- [2] G. Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, Dover Publications, New York, NY, 1955. [6](#)
- [3] K. K. Nambiar, *Ackermann Functions and Transfinite Ordinals*, Computers and Mathematics with Applications **8** (1995), no. 6, 51–53. [10](#)
- [4] K. K. Nambiar, *Real Set Theory*, Computers and Mathematics with Applications **38** (1999), no. 7-8, 167–171. [12](#)

EXERCISES

EXERCISE 1. Evaluate $2 \otimes^3 4$ and compare it with 136×2^{256} , the number of electrons in the universe, as given by Eddington.

EXERCISE 2. Prove or disprove that $2 \otimes^3 n$ is a prime, for all positive integer values of n .

EXERCISE 3. Show that $m \otimes^k n$ gives the solution to Ackermann Equations.

Solutions to Exercises

Exercise 1. A detailed evaluation shows that

$$\begin{aligned}2 \otimes^3 4 &= 2 \otimes^2 [2 \otimes^2 [2 \otimes^2 2]] \\ &= 2 \otimes^2 [2 \otimes^2 4] \\ &= 2 \otimes^2 65536 \\ &= 2^{2^{\dots^2}},\end{aligned}$$

where the total number of 2's tilting forward is 65536. Obviously, there is no comparison between $2 \otimes^3 4$ and the number of electrons in the universe.

Exercise 1

Exercise 2. It took a fortunate Euler to disprove Fermat's conjecture that

$$2^{2^n} + 1$$

is a prime, for all n . The following shows what we are up against.

$$\begin{aligned} 2 \otimes^3 1 + 1 &= 2 + 1 \\ &= 3, \text{ a prime} \end{aligned}$$

$$\begin{aligned} 2 \otimes^3 2 + 1 &= 2^2 + 1 \\ &= 4 + 1 \\ &= 5, \text{ a prime} \end{aligned}$$

$$\begin{aligned} 2 \otimes^3 3 + 1 &= 2^{2^{2^2}} + 1 \\ &= 65536 + 1 \\ &= 65537, \text{ a prime} \end{aligned}$$

Since,

$$2 \otimes^3 4 + 1$$

is unimaginably large, as shown earlier, we will never be able to decide for

sure whether it has a factor. A similar statement applies to

$$2 \otimes^3 n + 1,$$

for $n > 4$. In short, we cannot decide the conjecture.

Exercise 2

Exercise 3. Ackermann functions are defined by the equations,

$$A(0, n) = mn$$

$$A(k, 1) = m$$

$$A(k, n) = A[k - 1, A(k, n - 1)].$$

Comparing this with the definition of \otimes^k shows that there is virtually no difference between the two.

End of Document

Exercise 3