Derivation of Continuum Hypothesis from Axiom of Combinatorial Sets

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In set theory [1], two sets are considered to have the same cardinality, if a one-to-one correspondence can be set up between them. Cantor has shown that the powerset of a set can never be put into one-to-one correspondence with the original set itself, even when the given set is infinite. What this means is that we can go on taking powerset of powersets to produce larger and larger sets. Thus starting with $\aleph_0$, the set of natural numbers, we can repeatedly take powersets and end up with an infinite sequence of infinite sets of ever increasing size. If we use the notation $2^{\aleph_0}$ for the powerset of $\aleph_0$, the natural question that we face is the following: If $\aleph_1$ is the bigger infinity next to $\aleph_0$, is it the same as $2^{\aleph_0}$? In other words, is there an infinity $\aleph_1$, that is larger than $\aleph_0$, but less than $2^{\aleph_0}$? Cantor's guess about the answer to this question is called the Continuum Hypothesis (CH), an issue that has occupied the minds of mathematicians for the whole of last century and continues to do so:

Continuum Hypothesis:

$$\aleph_1 = 2^{\aleph_0}.$$

If we represent by

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$$

the consecutive transfinite cardinals of Cantor, the generalized version of the continuum hypothesis (GCH) can be stated as

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}.$$

In this note, however, we will restrict ourselves to the derivation of the continuum hypothesis, with the understanding that the derivation of GCH will not be substantially different from that of CH.

Producing $\aleph_1$ from $\aleph_0$. Halmos explains [4, p. 77] the generation of $\aleph_1$ from ω, the ordinal corresponding to $\aleph_0$, as given below.

> ... In this way we get successively ω, $\omega 2$, $\omega 3$, $\omega 4$, $\cdots$. An application of the axiom of substitution yields something that follows them all in the same sense in which ω follows the natural numbers; that something

is ω^2. After that the whole thing starts over again: ω^2+1, ω^2+2, $\cdots$, $\omega^2+\omega$, $\omega^2+\omega+1$, $\omega^2+\omega+2$, $\cdots$, $\omega^2+\omega 2$, $\omega^2+\omega 2+1$, $\cdots$, $\omega^2+\omega 3$, $\cdots$, $\omega^2+\omega 4$, $\cdots$, $\omega^2 2$, $\cdots$, $\omega^2 3$, $\cdots$, ω^3, $\cdots$, ω^4, $\cdots$, ω^ω, $\cdots$, $\omega^{(\omega^\omega)}$, $\cdots$, $\omega^{(\omega^{(\omega^\omega)})}$, $\cdots\ \cdots$. The next one after all this is ϵ_0; then come ϵ_0+1, ϵ_0+2, $\cdots$, $\epsilon_0+\omega$, $\cdots$, $\epsilon_0+\omega 2$, $\cdots$, $\epsilon_0+\omega^2$, $\cdots$, $\epsilon_0+\omega^\omega$, $\cdots$, $\epsilon_0 2$, $\cdots$, $\epsilon_0\omega$, $\cdots$, $\epsilon_0\omega^\omega$, $\cdots$, ϵ_0^2, $\cdots\ \cdots\ \cdots$.

Here, Cantor tells us that just as we get $\aleph_0$ by writing the natural numbers as an increasing infinite sequence, $\aleph_1$ can also be obtained as an increasing infinite sequence of counting numbers. The complex notations that we see in this quote are clever artifices to shorten the enormously long sequence of counting numbers that we have to deal with. Incidentally, the quote gives the most sophisticated use of ellipses that we are aware of.

A Candidate Axiom. If k is an ordinal, we write

$$\binom{\aleph_0}{k}$$

for the cardinality of the set of all subsets of $\aleph_0$ with cardinality as that of k. With this notation, the axiom we are interested in can be stated as a simple equation.

Axiom of Combinatorial Sets:

$$\aleph_1 = \binom{\aleph_0}{\aleph_0}.$$

It turns out that if we accept this axiom, the derivation of the continuum hypothesis becomes very straightforward. We need the following ad hoc definitions for the derivation.

Even-Set: An *infinite* set of positive even integers. Example:

$$\{4, 10, 16, 22, 28, \ldots\}.$$

Note that every even-set corresponds to an infinite set of integers, obtained by dividing each number in the set by 2. For our example, the infinite set is

$$\{2, 5, 8, 11, 14, \ldots\}.$$

Odd-Set: A *finite* set of positive even integers, along with all the odd integers above the largest even integer in the set. Example:

$$\{4, 10, 14, 20, 21, 23, 25, 27, \ldots\}.$$

Note that every odd-set corresponds to a finite set of integers, for our example, the finite set is

$$\{2, 5, 7, 10\}.$$

Derivation of Continuum Hypothesis. Clearly, $\binom{\aleph_0}{\aleph_0}$ is a subset of the powerset $2^{\aleph_0}$, and hence, $\binom{\aleph_0}{\aleph_0} \leq 2^{\aleph_0}$. It is visibly clear from the definitions of even-sets and odd-sets that they are elements of $\binom{\aleph_0}{\aleph_0}$, and hence, $\binom{\aleph_0}{\aleph_0} \geq 2^{\aleph_0}$. If we use the axiom of combinatorial sets, continuum hypothesis immediately follows.

Conclusion. It is known [2, 3] that the addition of an axiom is necessary to derive the continuum hypothesis in Zermelo-Fraenkel set theory. Hence, the question we have to answer after reading this note is: Is the Axiom of Combinatorial Sets the right axiom we ought to have in set theory?

References

[1] G. Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, Dover Publications, New York, NY, 1955.

[2] P. Cohen, *Set Theory and the Continuum Hypothesis*, W.A. Benjamin Inc., New York, 1966.

[3] K. Gödel, *The Consistency of Continuum Hypothesis*, Princeton University Press, Princeton, NJ, 1940.

[4] P. R. Halmos, *Naive Set theory*, D. Van Nostrand Company, New York, NY, 1960.