Hall’s Theorem and Compound Matrices

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Abstract—Compound matrices which list the entire set of matchings in a bigraph are used to prove the theorem of Hall.

Keywords—Compound matrices; Hall’s theorem; Matchings.

1. INTRODUCTION

The purpose of this paper is to show that the theory of compound matrices [1] provides a convenient and efficient notation for stating and proving Hall’s theorem [2]. The theorem states that in a bigraph with vertex sets \( V_1 \) and \( V_2 \), a complete matching of \( V_1 \) into \( V_2 \) exists, if and only if, every subset of \( k \) vertices in \( V_1 \) is adjacent to at least \( k \) vertices in \( V_2 \), for all values of \( k \). In the bigraph, we assume \( |V_1| = m, |V_2| = n \) and \( m \leq n \).

2. COMPOUND MATRICES

From a matrix \( A \) of order \( m \times n \), when the minors of order \( k \) are arranged in the lexical order, the resulting \( \binom{m}{k} \times \binom{n}{k} \) matrix is called the \( k \)th compound of \( A \) and written as \( A^{(k)} \). See below where the adjacency matrix \( A \) of the bigraph in Fig. 1 is taken as an example.

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
12 & 13 & 14 & 23 & 24 & 34 \\
12 & a_{1b_2} & a_{1b_3} & a_{1b_4} & a_{2b_3} & a_{2b_4} & a_{2b_4} & 0 \\
13 & 0 & a_{1c_3} & a_{1c_4} & a_{2c_3} & a_{2c_4} & a_{2c_4} & 0 \\
23 & 0 & 0 & 0 & b_{2c_3} & b_{2c_4} & b_{3c_4} & b_{4c_3}
\end{pmatrix}
\]

Fig. 1
\[
A^{(3)} = \begin{pmatrix}
123 & 124 & 134 & 234
\end{pmatrix}
\begin{pmatrix}
a_1 b_2 c_3 & a_1 b_2 c_4 & a_1 b_3 c_4 & -a_1 b_4 c_3 \\
a_2 b_3 c_4 & -a_2 b_3 c_4 & -a_2 b_4 c_3 & a_2 b_4 c_4
\end{pmatrix}
\]

Given below are two theorems on compound matrices that we need for proving Hall’s theorem. Here \(P\) and \(Q\) are assumed to be conformable and \(R\) is of order \(r\).

\[
(PQ)^{(k)} = P^{(k)}Q^{(k)}
\]

\[
|R^{(k)}| = |R^{(r-1)}|
\]

With these preliminaries we can now deal with Hall’s theorem.

3. PROOF OF HALL’S THEOREM

It is easy to state Hall’s theorem in terms of the adjacency matrix \(A\) of the bigraph.

**Hall’s Theorem.** For some \(k\), a row of \(A^{(k)} = 0\), if and only if, \(A^{(m)} = 0\).

**Proof.** We carry out the proof in two parts.

Part 1.

For some \(k\), a row of \(A^{(k)} = 0 \Rightarrow |A^{(k)}A_T^{(k)}| = 0
\]

\[
\Rightarrow |(AA_T)^{(k)}| = 0
\]

\[
\Rightarrow |AA_T|^{(m-1)} = 0
\]

\[
\Rightarrow |AA_T| = 0
\]

\[
\Rightarrow A^{(m)}A_T^{(m)} = 0
\]

\[
\Rightarrow A^{(m)} = 0
\]

Part 2.

\(A^{(m)} = 0 \Rightarrow \) For \(k = m\), the row matrix \(A^{(k)} = 0
\]

\[
\Rightarrow \) For some \(k\), a row of \(A^{(k)} = 0
\]

Hall’s theorem immediately follows.

4. CONCLUSION

Since the terms in a determinant represent matchings, it is not surprising that compound matrices are useful in analyzing matching problems. It is easy to see that the compound matrices give the *entire* set of matchings in a bigraph.

REFERENCES