MATRICES WITH ELEMENTS FROM A DIVISION RING

K. K. Nambiar
School of Computer and Systems Sciences
Jawaharlal Nehru University, New Delhi 110067, India

(Received June 1995)

Abstract—Closure of a regular matrix is defined and used to enumerate paths in a graph. It is shown that closure can be utilized to get the generator equations of a group.

Keywords: Path enumeration; Group generators; Word space.

1. INTRODUCTION

For the graph shown in Fig. 1 the adjacency matrix[1] can be written as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and if we take the inverse of the matrix $(\mathbf{I} - x\mathbf{A})$, where $\mathbf{I}$ is the unit matrix, we can get the number of paths between any pair of nodes. Since the (1,2) element in the inverse

$$(\mathbf{I} - x\mathbf{A})^{-1} = \begin{bmatrix} (1-x-x^2)^{-1} & x(1-x-x^2)^{-1} \\ x(1-x-x^2)^{-1} & (1-x)(1-x-x^2)^{-1} \end{bmatrix}$$

is the generator function for the Fibonacci sequence[2], we can conclude that the number of paths of different lengths from node 1 to node 2 is given by the Fibonacci numbers. It is shown in the sequel that instead of just the number of paths, we can obtain the entire set of paths itself if we make use of matrices with elements from a division ring.

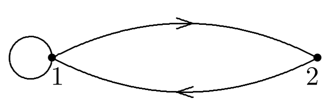

Fig. 1

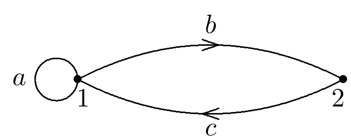

Fig. 2

2. DEFINITIONS

1. *Regular algebra*: Same as division ring.
2. *Slippery element*: An element of a regular algebra which commutes with all the other elements.
3. *Regular matrix*: A matrix with elements from a division ring.
4. *Regular inverse*: If the matrix $\mathbf{A}$ in the partitioned form is

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

with $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ being square, the regular inverse is given by the recursive definition

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}_{22}^{-1} \\ -\mathbf{B}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{B}_{22}^{-1} \end{bmatrix}$$

where $\mathbf{B}_{22} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$. By actual multiplication, it can be verified that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

5. *Regular closure*: The symbol for closure of $\mathbf{A}$ is $\mathbf{A}^*$. It is defined as $(\mathbf{I} - \mathbf{A})^{-1}$. Our usage of $*$ here is only a slight generalization of the use of $*$ for regular expressions in automata theory and hence in no way should create any confusion. $\mathbf{A}^*$ can be defined recursively as

$$\mathbf{A}^* = \begin{bmatrix} \mathbf{A}_{11}^* + \mathbf{A}_{11}^*\mathbf{A}_{12}\mathbf{B}_{22}^*\mathbf{A}_{21}\mathbf{A}_{11}^* & \mathbf{A}_{11}^*\mathbf{A}_{12}\mathbf{B}_{22}^* \\ \mathbf{B}_{22}^*\mathbf{A}_{21}\mathbf{A}_{11}^* & \mathbf{B}_{22}^* \end{bmatrix}$$

where $\mathbf{B}_{22} = \mathbf{A}_{22} + \mathbf{A}_{21}\mathbf{A}_{11}^*\mathbf{A}_{12}$.

3. EXAMPLES

Example 1: For the graph shown in Fig. 2 the adjacency matrix is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$$

and

$$\mathbf{A}^* = \begin{bmatrix} a^* + a^*b(ca^*b)^*ca^* & a^*b(ca^*b)^* \\ (ca^*b)^*ca^* & (ca^*b)^* \end{bmatrix}.$$

The elements of $\mathbf{A}^*$ gives all the different paths between the nodes. If we are interested in isolating paths of different lengths we may use the labels za, zb, and zc instead of a, b, and c, where z is a slippery element. If the graph respresents an automaton with 1 as initial node and 2 as final node, then the language of the automaton is given by the (1,2) element $a^*b(ca^*b)^*$. Here it happens that there are no duplication of strings, but in general it can happen, in which case the duplication should be eliminated.

Example 2: The adjacency matrix of the Cayley graph[1] of the dihedral group D_2 can be written as

$$\mathbf{A} = \begin{bmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & b & 0 & a \\ b & 0 & a & 0 \end{bmatrix}$$

and

$$\mathbf{A}^* = \begin{bmatrix} p & q & pr+qs & ps+qr \\ q & p & ps+qr & pr+qs \\ pr+qs & ps+qr & p & q \\ ps+qr & pr+qs & q & p \end{bmatrix}$$

where

$$p = [b(aa)^*b + (a + b(aa)^*ab)(b(aa)^*b)^*(a + b(aa)^*ab)]^*$$
$$q = [b(aa)^*b + (a + b(aa)^*ab)(b(aa)^*b)^*(a + b(aa)^*ab)]^*(a + b(aa)^*ab)(b(aa)^*b)^*$$
$$r = b(aa)^*a$$
$$s = b(aa)^*.$$

Note that the diagonal elements here are equal. It is easy to see that this has to be so, for a Caley graph. The generator equations of the group can be obtained by equating every term in the diagonal element and also every starred term within the term to 1.
Putting $aa = 1$:

$$p \Rightarrow [bb + (a + bab)(bb)^*(a + bab)]^*$$

Putting $bb = 1$:

$$\begin{aligned} p &\Rightarrow [(a + bab)(a + bab)]^* \\ &\Rightarrow [aa + abab + baba + babbab]^* \\ &\Rightarrow [abab + baba]^* \end{aligned}$$

Putting $abab = 1$:

$$p \Rightarrow 1$$

Note that $abab = 1$ implies $baba = 1$. Thus we get the equations for the dihedral group as $aa = bb = abab = 1$.

4. CONCLUSION

The legitimacy of our computations is based on a structure that we could call *word space*. From the definition below it should be clear that every vector space is a word space.

$\mathcal{W}$: a commutative group, $\mathbf{w} \in \mathcal{W}$.

$\mathcal{R}$: a regular algebra, $c \in \mathcal{R}$.

Scalar multiplication: $\mathcal{R} \times \mathcal{W} \to \mathcal{W}$ and $\mathcal{W} \times \mathcal{R} \to \mathcal{W}$.

With these notations, $\mathcal{W}$ is a word space, if it satisfies:

$$\begin{aligned} &\mathbf{w}1 = \mathbf{w} && 1\mathbf{w} = \mathbf{w} \\ &(\mathbf{w}c_1)c_2 = \mathbf{w}(c_1c_2) && c_1(c_2\mathbf{w}) = (c_1c_2)\mathbf{w} \\ &\mathbf{w}(c_1 + c_2) = \mathbf{w}c_1 + \mathbf{w}c_2 && (c_1 + c_2)\mathbf{w} = c_1\mathbf{w} + c_2\mathbf{w} \\ &(\mathbf{w}_1 + \mathbf{w}_2)c = \mathbf{w}_1c + \mathbf{w}_2c && c(\mathbf{w}_1 + \mathbf{w}_2) = c\mathbf{w}_1 + c\mathbf{w}_2 \end{aligned}$$

It should be interesting to isolate those properties of the vector space which are retained by the word space, especially regarding the eigenvalues and eigenvectors.

References

1. O. Ore, *Theory of Graphs,* Ameican Mathematical Society, Providence, RI, (1962).
2. G. E. Andrews, *Number Theory,* W. B. Saunders Company, Philadelphia, PA, (1971).