A GRAPH-THEORETIC PROOF OF ARROW'S DICTATOR THEOREM

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(Submitted May 1992)

Communicated by E. Y. Rodin

Abstract–The preference of a voter is represented by a preference graph and an elementary proof of the Dictator Theorem is given using preference matrices.

1. INTRODUCTION

Arrow's Dictator Theorem states that if the input to a voting system is the preference of each voter for the candidates, then there is no reasonable way to assign a preference appropriate for the society as a whole. Various proofs of the theorem have appeared in the last four decades[**1-3**], we consider the proof given here to be particularly simple.

There are two types of directed graphs we should define before taking up the proof: *preference graphs* and *nonpreference graphs*. A nonpreference graph is one which is complete and transitive. A graph is complete if there is atleast one edge, in either direction, between any pair of nodes(not necessarily distinct). A graph is transitive if, whenever edges (i, j) and (j, k) exist in the graph, then (i, k) also exists. It is well known[**4**] that a nonpreference graph arranges the nodes of a graph in a linear order, possibly some nodes clubbed together, and it can be used to represent the preference of a voter for the candidates. Obviously the complement of a nonpreference graph can also be used to represent the preference of a voter. We call this complement, preference graph and its adjacency matrix, *preference matrix*. It is easy to see that the preference graph satisfies asymmetry and transitivity. The notations used in the proof are given below :

1. m : the total number of candidates $C_1, C_2, \cdots, C_m$
2. n : the total number of voters $V_1, V_2, \cdots, V_n$
3. $\mathbf{V}_k = [v_{ij}^k]$: the preference matrix of order $m \times m$ giving the preference of the voter V_k. $v_{ij}^k = 0$ means that the voter V_k does not prefer C_i over C_j. We use $\mathbf{0}$, boldface zero, to represent nonpreference of a set of voters. $v_{ij}^k = 1$ means that the voter V_k prefers C_i over C_j. We use $\mathbf{1}$, boldface one, to represent preference of a set of voters. $v_{ij}^k = *$ means unspecified preference of the voter V_k. We use $\star$, a star, to reperesent unspecified preference of a set of voters
4. $\mathbf{S} = [s_{ij}]$: the preference matrix of order $m \times m$ giving the preference of the society as a whole
5. *Voting System* : a function $\mathbf{F}(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_n) = \mathbf{S}$
6. *Dictator Function* : $D_n^k(x_1, x_2, \cdots, x_n) = x_k$, also called *projection function*.

2. AXIOMS, THEOREM, AND PROOF

Two axioms are assumed by Arrow in the derivation of the Dictator Theorem.

1. *Axiom of Independence* : $s_{ij} = f_{ij}(v_{ij}^1, v_{ij}^2, \cdots, v_{ij}^n)$ for $i \neq j$ and $s_{ii} = 0$.

The axiom states that s_{ij} is a function of v_{ij}^k's only and nothing else.

2. *Axiom of Unanimity* : $f_{ij}(0, 0, \cdots, 0) = 0$ and $f_{ij}(1, 1, \cdots, 1) = 1$.

The axiom states that if all the voters, without exception, vote one way then the voting system also votes the same way.

Dictator Theorem : $f_{ij}(x_1, x_2, \cdots, x_n) = D_n^d(x_1, x_2, \cdots, x_n) = x_d$. *In other words* $\mathbf{S} = \mathbf{V}_d$.

Proof : Define

$$h = \min_{ij} \left\{ \Sigma_{k=1}^n x_k \;\middle|\; f_{ij}(x_1, x_2, \cdots, x_n) = 1 \right\}$$

Note that $m(m-1)2^n$ values of f_{ij} are to be inspected before we can obtain the value of h. We want to show that $h = 1$.

$f_{ij}(\mathbf{1}, \mathbf{0}, \mathbf{0}) = 0$ and $f_{jk}(\mathbf{1}, \mathbf{0}, \mathbf{0}) = 0$

$\Rightarrow f_{ik}(\star, \star, \mathbf{0}) = 0$ since nonpreference graphs are transitive

$\Rightarrow f_{ik}(\mathbf{1}, \mathbf{1}, \mathbf{0}) = 0$

Taking the contrapositive of the above argument

$f_{ik}(\mathbf{1}, \mathbf{1}, \mathbf{0}) = 1$

$\Rightarrow f_{ij}(\mathbf{1}, \mathbf{0}, \mathbf{0}) = 1$ or $f_{jk}(\mathbf{0}, \mathbf{1}, \mathbf{0}) = 1$

It immediately follows that $h = 1$. Note that h cannot be zero because of the Unanimity axiom.

Without loss of generality we may assume $f_{ab}(1, \mathbf{0}) = 1$, the position at which 1 occurs in f_{ab} is of no concern to us. Here C_a and C_b are two specific candidates. Now,

$f_{ia}(1, \mathbf{1}) = 1$ and $f_{ab}(1, \mathbf{0}) = 1$

$\Rightarrow f_{ib}(1, \star) = 1$ since preference graphs are transitive, and

$f_{ib}(1, \star) = 1$ and $f_{bj}(1, \mathbf{1}) = 1$

$\Rightarrow f_{ij}(1, \star) = 1$ since preference graphs are transitive

$\Rightarrow f_{ji}(0, \star) = 0$ since preference graphs are asymmetric

$\Rightarrow f_{ij}(x_1, \star) = x_1$

Dictator Theorem immediately follows.

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